VARIATIONAL METHOD OF CONSTRUCTING SUBSONIC AIRFOIL PROFILES

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UDC 533.6.011

A method based on the descripition of flows by a functional which has the equations of ideal gas flow as its extremes has been developed to refine the solutions of problems involving the aerodynamic design of airfoil profiles. Study of the behavior of the second variation of the functional made it possible to identify its extremum, which in turn permits the calculation of mixed flows about airfoils (including transonic flows). The stream function through which all of the quantities in the functional are expressed is approximated by an expansion in a basis consisting of eigenfunctions of the Laplace operator. The problem is solved by defining the extremum in the finite-dimensional space of the weight factors of the basis. The value of the functional at the running point of the space is calculated using an approximation of the integrand by Hermite polynomials.

The solution of the problem of constructing critical airfoil profiles is equivalent to the construction of a streamline connecting specified sections of a boundary. The congruence theorem [1] makes it possible to change over from the construction of a free streamline with a local Mach number M = 1 to the problem of the maximum of the cross-sectional area of of the equations of gasdynamics. These constraints can be satisfied with minimization of the functional

$$I = \iint\limits_{\Omega} (p + \rho q^2)\, d\omega. \tag{1}$$

It is known [2] that the variational equations for (1) are the equations of plane flows:

$$\mathbf{q} \times (\nabla \times \mathbf{q}) = -\frac{1}{2}(a^2 - q^2)\nabla \ln f, \quad \nabla(\rho \mathbf{q}) = 0, \quad p = f\rho^{\gamma}, \quad \ln f = \frac{s}{c_V} \tag{2}$$

($\mathbf{q} \cdot \nabla f = 0$ is the adiabaticity condition).

The Bernoulli equation has the form

$$\frac{1}{2} q^2 + \frac{\gamma}{\gamma - 1}\frac{p}{\rho} = \frac{1}{2} a^2.$$

We used the following notation in (2): $\mathbf{q} = (u, v)$ is the velocity vector; p is pressure; ρ is density; a is sonic velocity; s is entropy; $\gamma = c_p/c_V$ is the adiabatic exponent; c_p and c_V are the heat capacities of the gas at constant pressure and volume; f is a function of ψ, which is a constant in the case of steady flow and the absence of vorticity at infinity if there are no shock waves. Written in terms of the stream function $\psi(x, y)$, Eq. (1) appears as

$$I = \iint\limits_{\Omega} P(\psi, \nabla\psi)\, d\omega, \tag{3}$$

where

$$P(\psi, \nabla\psi) = 2\gamma \left[\frac{2\gamma}{\gamma - 1}\right]^{1/(\gamma-1)} (p + \rho q^2);$$

$$p = f(\psi)^{-1/(\gamma-1)} \left[\frac{\gamma - 1}{2\gamma}(1 - q^2)\right]^{\gamma/(\gamma-1)};$$

$$\rho = f(\psi)^{-1/(\gamma-1)} \left[\frac{\gamma - 1}{2\gamma}(1 - q^2)\right]^{1/(\gamma-1)}; \quad (1 - q^2)^{2/(\gamma-1)} q^2 = (\nabla\psi)^2.$$

If the Euler–Lagrange equations for (1) are to coincide with (2), it is necessary that

$$\int\limits_{\partial\Omega} \frac{\partial P}{\partial \nabla\psi} \mathbf{n} \delta\psi\, dl = 0.$$

Novosibirsk. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, No. 4, pp. 90-93, July-August, 1992. Original article submitted June 21, 1991; revision submitted July 11, 1991.

 0021-8944/92/3304-0558$12.50

This is valid for the boundary conditions in the problem of flow about an airfoil, since it reduces to the condition

$$\int_{\partial\Omega} (-v, u)\,\mathbf{n}\delta\psi\, dl = 0,$$

which is the difference in circulation on the external and internal boundaries. The second variation of (1) has the form

$$\delta^2 I = 2\gamma \iint_{\Omega} \Big[-\frac{1}{\gamma-1} f^{-\gamma/(\gamma-1)} \frac{\partial f}{\partial\psi}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\frac{1}{2\gamma}(1-q^2)^{\gamma/(\gamma-1)}\frac{\partial}{\partial\psi}\ln f\right) - $$
$$ - f^{1/(1-\gamma)}\frac{1}{2\gamma}(1-q^2)^{\gamma/(\gamma-1)}\frac{\partial^2}{\partial\psi^2}\ln f - \nabla\left(f^{-1/(\gamma-1)}\frac{\partial}{\partial\nabla\psi}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{1}{2\gamma}\,(1-q^2)^{\gamma/(\gamma-1)}\frac{\partial}{\partial\psi}\ln f\right)\right)\Big]\,\delta^2\psi\, d\omega. \tag{4}$$

In the above expressions, $\partial/\partial\nabla\psi \equiv (\partial/\partial\psi_x,\ \partial/\partial\psi_y)$, $\psi_x = \partial\psi/\partial x$, $\psi_y = \partial\psi/\partial y$. For flows with $f(\psi)$ = const, (4) reduces to

$$\delta^2 I = 2\gamma f^{-1/(\gamma-1)} \iint_{\Omega} \Delta\frac{1}{\rho}\,\delta^2\psi\, d\omega.$$

Alternatively, if we omit the positive constants we obtain

$$\delta^2 I \sim -\iint_{\Omega} (\nabla\mathbf{q})^2\,\delta^2\psi\, d\omega + \int_{\partial\Omega} \nabla\mathbf{q}\mathbf{n}\delta^2\psi\, dl \tag{5}$$

($d\Omega$ is the boundary of the region of integration Ω). The second term in (5) is equal to zero, since $\nabla\mathbf{q}$ is directed along a tangent to the contour and $\mathbf{q}$ = const in the undisturbed flow. Thus, it is necessary to minimize the convex functional.

Let the area functional

$$I_S = S^{-1}(F), \tag{6}$$

where $y = F(x, \mathbf{p})$ is the function giving the contour. It consists of cubic splines for the top and bottom contours of the profile. These splines are constructed from the nodes that specify them.

The solution of the projection problem is obtained by determining the extremum of the combined functional J. This functional consists of (1) and (6) in the finite-dimensional space of the weight factors of the basis used to represent the stream function and parameters $\mathbf{p}$ that give the profile.

In the present case, the stream function is approximated by an expansion in a basis consisting of eigenfunctions of the Laplace operator in the region Ω. The parameters $\mathbf{p}$ are the ordinates of the nodes which give the contour of the airfoil except for the fixed tip. The latter consists of a circle arc and the sections immediately above and below the trailing edge.

If the flow moves past the airfoil at the angle of attack α with the velocity $\mathbf{q}_\infty$, then with allowance for the asymptotic solution in [3] we can determine the stream function on the external boundary from the formulas

$$\begin{gathered}
\psi(x, y) = \rho_\infty\left[u_\infty y - v_\infty x - \frac{k\Gamma}{2\pi}\left(\int_0^y f_1(x, \xi)\,d\xi + \int_0^x f_2(\eta, y)\,d\eta\right)\right],\\
\int_0^y f_1(x, \xi)\,d\xi = \frac{1}{C_1}\left[\frac{E_1}{2}\ln(\xi^2 + D_1\xi + B_1) + \frac{2A_1 - E_1D_1}{\sqrt{4B_1 - D_1^2}}\operatorname{arctg}\frac{2\xi + D_1}{\sqrt{4B_1 - D_1^2}}\right]\Bigg|_0^y,\\
\int_0^x f_2(\eta, y)\,d\eta = \frac{1}{C_2}\left[\frac{E_2}{2}\ln(\eta^2 + D_2\eta + B_2) + \frac{2A_2 - E_2D_2}{\sqrt{4B_2 - D_2^2}}\operatorname{arctg}\frac{2\eta + D_2}{\sqrt{4B_2 - D_2^2}}\right]\Bigg|_0^x,\\
A_1 = -x\sin\alpha,\ A_2 = y\sin\alpha,\ E_1 = E_2 = \cos\alpha,\\
C_1 = \sin^2\alpha + k^2\cos^2\alpha,\ C_2 = \cos^2\alpha + k^2\sin^2\alpha,\\
B_1 = x^2(\cos^2\alpha + k^2\sin^2\alpha)/C_1,\ B_2 = y^2(\sin^2\alpha + k^2\cos^2\alpha)/C_2,\\
D_i = 2A_iE_i\mathrm{M}_\infty^2/C_i,\quad i = 1, 2,\quad k^2 = 1 - \mathrm{M}_\infty^2
\end{gathered} \tag{7}$$

(Γ is circulation).

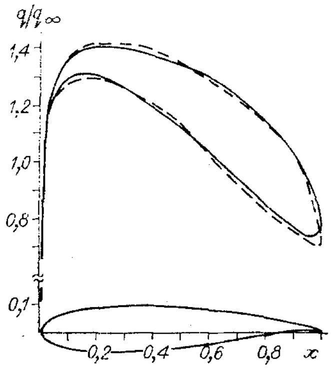

Fig. 1

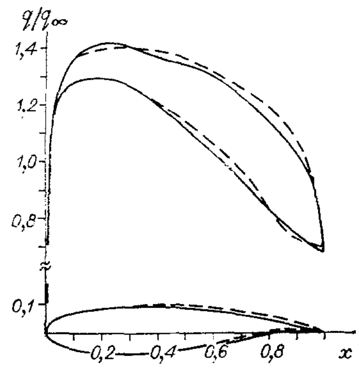

Fig. 2

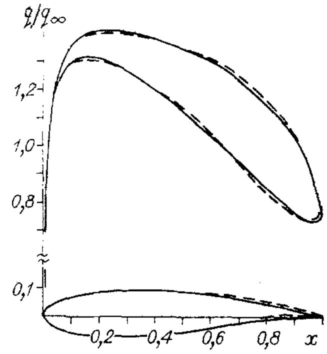

Fig. 3

As the initial airfoil, we take an airfoil with a fairly high critical Mach number (M_∞ = 0.68). This airfoil was designed by the method of constructing the quasi-solution of the inverse boundary-value problem for the Chaplygin gas model.* The angle at the trailing edge of this and all subsequent airfoils is equal to zero. We minimized functional (1) to calculate flow about the given profile within the framework of the equations of gasdynamics. Figure 1 shows the distributions of the corresponding velocities about the contour for different models. Here, the solid line corresponds to the Chaplygin gas (C_y = 0.383, C_x = −0.001), while the dashed line corresponds to an ideal gas (C_y = 0.48, $C_x \sim 10^{-5}$). The difference of C_x from zero is related to the computing error.

The above formulation of the design problem contains geometric constraints on the airfoil

$$x_k - x_0 = L,\ L = \text{const},\ F(x_0, \mathbf{p}) = 0,\ F(x_k, \mathbf{p}) = 0, \qquad (8)$$
$$F_x(x_0, \mathbf{p}) = \infty,\ F_x(x_k, \mathbf{p}) \geqslant \operatorname{tg}\theta_0,\ -\pi/2 \leqslant \theta_0 < 0$$

and aerodynamic constraints on the flow

$$M_\infty = M_\infty^0, \quad C_y \geqslant C_y^0, \quad M_\Omega \leqslant 1. \qquad (9)$$

The first optimization problem involves constructing a airfoil $F(x, \mathbf{p})$, that satisfies conditions (8) and (9) and has the maximum area. The second differs from the first in that, instead of flow functional (1), we took a functional corresponding to the Chaplygin gas model:

$$I = \iint_\Omega \sqrt{1 + \varphi_x^2 + \varphi_y^2}\, d\omega$$

[$\varphi(x, y)$ is the flow potential].

The conditions for φ on the external boundary are written on the basis of the same asymptote as was used for velocity (7). On the contour, instead of ψ = 0 we have $\partial\varphi/\partial n = 0$. In the case of isentropic flows, the Zhukov condition for airfoils with a sharp trailing edge is equivalent to equality of the values of velocity on the top and bottom sections of the airfoil near the edge. For a finite angle, the condition is equivalent to equality of the velocity to zero. This condition is introduced into J as a penalty function, and the circulation Γ calculated about the contour of the airfoil for each iteration with minimization of the functional is inserted into (7).

The dashed lines in Fig. 2 show the modified airfoil representing the solution of the first optimization problem with M_∞^0 = 0.68, C_y^0 = 0.4. Also shown is the distribution of velocity over the airfoil. The area increment $\Delta S \sim 6\%$.

Figure 3 shows the solution of the second optimization problem with $\Delta S \sim 3\%$. The dashed line shows the resulting airfoil. There is a small "ledge" corresponding to $M \approx 1$ in the velocity distributions on part of the top contours of the modified airfoils in Figs. 2 and 3. The situation M = const is not realized on any part of the free surface of these airfoils due to the lower bound for C_y in (9).

The results of the calculations confirm both the rigor of the proposed algorithm and airfoil design and the feasibility of using a Chaplygin gas as a model of the subsonic flow of a real gas in the design process. This means that more efficient design methods can be

*The authors are grateful to D. A. Fokin (of the N. G. Chebotarev Scientific-Research Institute of Mathematics and Mechanics, in Kazan') for providing the airfoil design and the velocity distribution on it.

developed, since the solutions of the equations of minimum surfaces can be represented in terms of analytic functions of a complex variable.

The author thanks A. F. Latypov for the advice given in the course of the investigation.

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NUMERICAL MODELING OF IMPULSIVE JETS OF A VISCOUS HEAT-CONDUCTING GAS

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Complete solution of the system of Navier–Stokes equations by the steady-state method with an implicit branching scheme for jets discharged from nozzles into a vacuum [1], submerged space [2], or companion flow [3] makes it possible to examine the subsonic sections of the flow and their effect on the structure of the jet as a whole. Additionally, the method proposed in [4, 5] makes it possible to calculate nonsteady processes. In the present study, we use this method to solve the problem of the discharge of a gas jet into a submerged space in the impulsive regime. This problem has application to both the formation of jets and the flow of quasisteady erosive jets.

1. Formulation of the Problem. We will examine a two-dimensional (axisymmetric) problem. Figure 1 presents a sketch of a region we are studying. It is bounded below by the axis of the jet OD. Line OA represents a sound outlet of radius r_e (or the edge of a supersonic nozzle). Above the edge of the nozzle, the boundary of the region of integration is a solid infinite 1 or finite 2 surface. As an alternative, AB may be a free boundary 3. The external boundary BC is located a distance from the axis such that the gas in the submerged space can be considered undisturbed. The boundary CD – where conditions corresponding to free discharge are established – is located a distance more than $20r_e$ from the nozzle edge.

The system of Navier–Stokes equations for a viscous compressible heat-conducting gas are written as follows in dimensionless form:

$$\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{v}{y}\right)=0,$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+(\gamma-1)\frac{\partial e}{\partial x}+(\gamma-1)\frac{e}{\rho}\frac{\partial\rho}{\partial x}=$$
$$=\frac{1}{\mathrm{Re}\,\rho}\left\{\frac{4}{3}\frac{\partial}{\partial x}\mu\frac{\partial u}{\partial x}+\frac{\partial}{\partial y}\mu\frac{\partial u}{\partial y}+\frac{\mu}{y}\frac{\partial u}{\partial y}+\frac{\partial}{\partial y}\mu\frac{\partial v}{\partial x}-\right.\quad(1.1)$$
$$\left.-\frac{2}{3}\frac{\partial}{\partial x}\mu\frac{\partial v}{\partial y}+\frac{\mu}{y}\frac{\partial v}{\partial x}-\frac{2}{3}\frac{1}{y}\frac{\partial}{\partial x}(\mu v)\right\},$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+(\gamma-1)\frac{\partial e}{\partial y}+(\gamma-1)\frac{e}{\rho}\frac{\partial\rho}{\partial y}=$$
$$=\frac{1}{\mathrm{Re}\,\rho}\left\{\frac{\partial}{\partial x}\mu\frac{\partial v}{\partial x}+\frac{4}{3}\left(\frac{\partial}{\partial y}\mu\frac{\partial v}{\partial y}+\frac{\mu}{y}\frac{\partial v}{\partial y}+\frac{\mu v}{y^2}\right)+\frac{\partial}{\partial x}\mu\frac{\partial u}{\partial y}-\frac{2}{3}\frac{\partial}{\partial y}\mu\frac{\partial u}{\partial x}-\frac{2}{3}\frac{v}{y}\frac{\partial\mu}{\partial y}\right\},$$

$$\frac{\partial e}{\partial t}+u\frac{\partial e}{\partial x}+v\frac{\partial e}{\partial y}+(\gamma-1)e\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{v}{y}\right)=\frac{1}{\mathrm{Re}\,\rho}\left\{\frac{\gamma}{\mathrm{Pr}}\left(\frac{\partial}{\partial x}\mu\frac{\partial e}{\partial x}+\frac{\partial}{\partial y}\mu\frac{\partial e}{\partial y}+\frac{\mu}{y}\frac{\partial e}{\partial y}\right)+\right.$$
$$\left.+\frac{4}{3}\mu\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2-\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right]+\mu\left[\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right]+\frac{4}{3}\frac{\mu v}{y}\left(\frac{v}{y}-\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\right)\right\}.$$

Novosibirsk. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, No. 4, pp. 93-99, July-August, 1992. Original article submitted May 5, 1991; revision submitted July 5, 1991.

0021-8944/92/3304-0561$12.50